



Time dependent energy balance model - a tutorial tool

Description of a web-based model used in various courses

Eigil Kaas*

Niels Bohr Institute, Copenhagen, Denmark

1 Introduction

These notes provide a description of a time dependent one-dimensional energy balance climate model. I.e. it is model which is designed to simulate the latitudinal and temporal evolution of the zonally averaged - or symmetric - surface temperature of the Earth. The model is symmetric around equator, and it is a so called aqua-planet model, which - in this context - means that the simulated surface temperature represents the average temperature of a layer of water. In its present configuration the model has no annual cycle.

Technically the model is set up as an Internet based application, where the user interactively can modify a number of input parameters.

When launched the model first performs a long so-called control simulation which is supposed to represent present day climate conditions. The final state of this simulation is then used as starting point for a second simulation of the same length. In this second run the model can be exposed to an external user defined (radiative) forcing in addition to the present day short and long wave forcings.

In the following, section 2 provides a brief introduction to the simplest types of energy balance models while section 3 describes the specific model used here. Section 4 is an overview of the experimental setup which is used when the model is run, and lists different input parameters to the model, which may be changed by the user, and section 5 describes practical details when

* *Corresponding author address:* Eigil Kaas, Niels Bohr Institute, Juliane Maries Vej 30, DK-2100 Copenhagen, Denmark. E-mail: kaas@gfy.ku.dk

running the model. Finally section 6 suggests a few exercises that can be used to understand the model and various basic concepts of climate research.

2 Simple energy balance models

Considering our entire planet over a long period the absorption of solar radiation must be balanced approximately by radiation emitted by the Earth system to space. Thus, in the simplest type of climate model one estimates a so called effective planetary black body emission temperature T_e , which balances the globally averaged absorbed solar radiation. Since the total area of the Earth is $4\pi a^2$, where a is the radius of the Earth, we therefore get that the total amount of energy emitted (i.e. lost) to space, the global average of the so-called outgoing long wave radiation OLR , is:

$$4\pi a^2 OLR = 4\pi a^2 \sigma T_e^4 \quad . \quad (1)$$

This is Stefan-Boltzmann's law giving the total radiation emitted from a perfect black body with temperature T_e and with an area of $4\pi a^2$. In (1) $\sigma = 5.67 \times 10^{-8} \text{W/m}^2/\text{K}^4$ is Stefan-Boltzmann's constant. Since the temperature of the Earth is low as compared to the sun, the radiation emitted by the Earth system is dominated by infrared wavelengths. This radiation is often referred to as long wave radiation as opposed to short wave (solar) radiation.

The total radial flux of solar radiation at the Earth's distance from the sun is the solar constant, $S \approx 1366 \text{W/m}^2$. Since the area of the Earth disc, as seen from the sun is πa^2 , the total downward component (as seen from the top of the Earth's atmosphere) of solar radiation is $\pi a^2 S$. A certain fraction, the planetary albedo α , of this short wave radiation is reflected back to space. Therefore the total amount of solar - i.e. short wave - radiation absorbed on the planet is:

$$E_{in} = \pi a^2 S(1 - \alpha) \quad . \quad (2)$$

The energy balance requirement $E_{in} - 2\pi a^2 OLR \approx 0$ gives the following time independent simple energy balance climate model:

$$\frac{S(1 - \alpha)}{4} - \sigma T_e^4 = 0 \quad . \quad (3)$$

Solving for T_e we get $T_e = 254\text{K}$, using a satellite based estimate of $\alpha = 0.31$.

Obviously, the Earth's effective temperature, i.e. solving (3), constitutes a poor model for the temperature near the Earth's surface: At present the area weighted arithmetic mean of the near surface global temperature T_s is about 288K. The main reason for the difference $\delta T = T_s - T_e$ is the atmosphere - or greenhouse - effect: The atmosphere contains clouds, greenhouse gases and particles which can absorb and emit infrared radiation. The radiation to space is on the average emitted from molecules and cloud droplets / ice crystals relatively high up in the atmosphere where the temperature is much lower than the temperature of the surface where most solar radiation is absorbed. This is the essence of the greenhouse effect: The average emission temperature for molecules emitting photons which escape to space without being re-absorbed is much lower than the surface temperature of the Earth. Photons emitted from the relatively warm surface will, with a high likelihood, be absorbed in the atmosphere, i.e. they will only rarely escape directly to space. In yet other words: the greenhouse effect act as an isolating

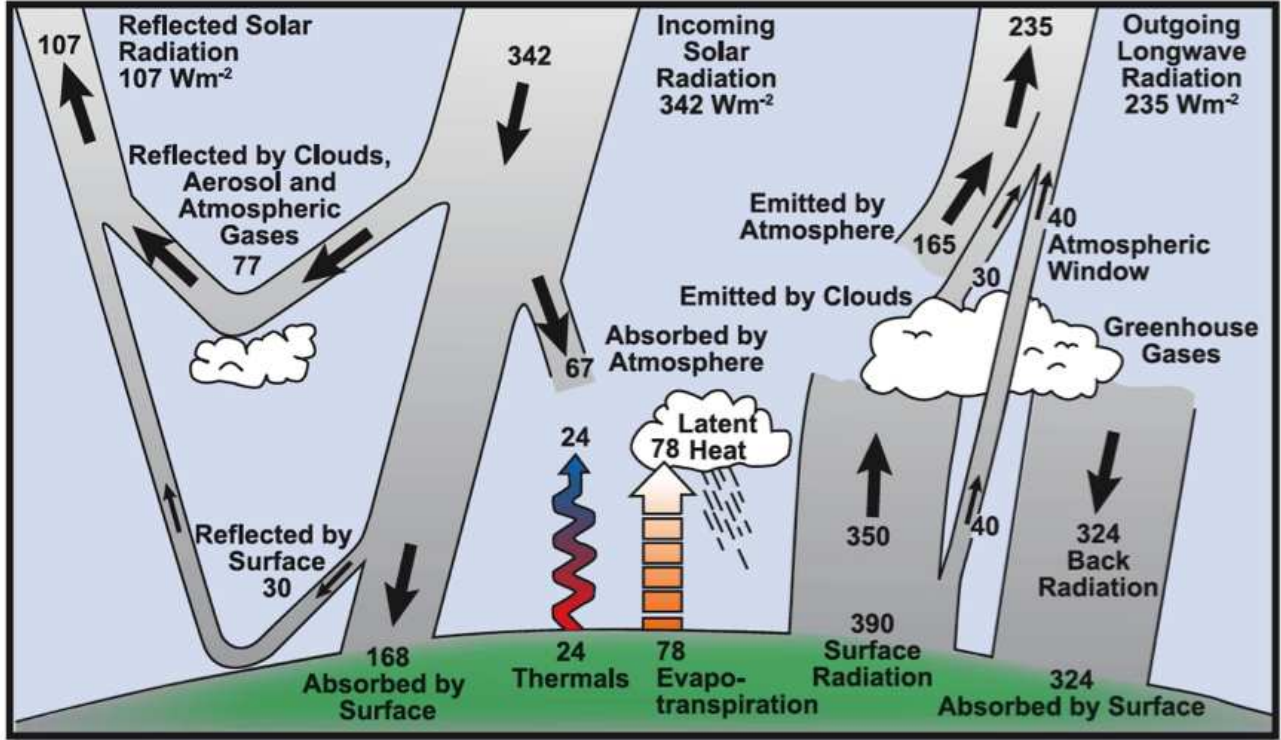


Figure 1: Estimated global averages of energy fluxes between, the surface, the atmosphere and space (From Kiehl and Trenberth (1997)).

blanket keeping the surface considerably warmer than it would be without the existence of the atmosphere.

The greenhouse effect is normally defined as the difference between the global averages of σT_s^4 and OLR . $\delta T = T_s - T_e$ is therefore a simple measure of the strength of the greenhouse effect. The stronger the greenhouse effect the larger δT , and thereby σT_s^4 if S and α are unchanged. We can therefore re-write (3) as follows, where the "unknown" variable is T_s ¹:

$$\frac{S(1 - \alpha)}{4} - \sigma(T_s - \delta T)^4 = 0 \quad . \quad (4)$$

It can be seen from Figure 1 that the largest contribution to the present day planetary albedo of $\alpha = 0.31$ is due to clouds, aerosols and gases². In addition to their direct influence, aerosols also have an indirect influence on α . The indirect effect is due to the role of some aerosols as cloud condensation nuclei. It is noted, that the present day surface contribution to α is less than half of that due to clouds. Furthermore, only a relatively small fraction of the present day surface contribution to α can be ascribed to ice and snow.

The present day greenhouse effect, corresponding to $\delta T = 34\text{K}$, is determined by the actual atmospheric state including its distribution of water vapour, well mixed greenhouse gases³, clouds

¹Note that (4) is constructed to lead to the present day T_s of 288 K when the present day values for α and δT , i.e. 0.31 and 34 K, respectively, are inserted.

²Some well mixed greenhouse gases, e.g. CO_2 , contribute slightly to α .

³Well mixed greenhouse gases, i.e. CO_2 , CH_4 , N_2O and CFC gases, survive so long time in the atmosphere that their concentration is almost uniform.

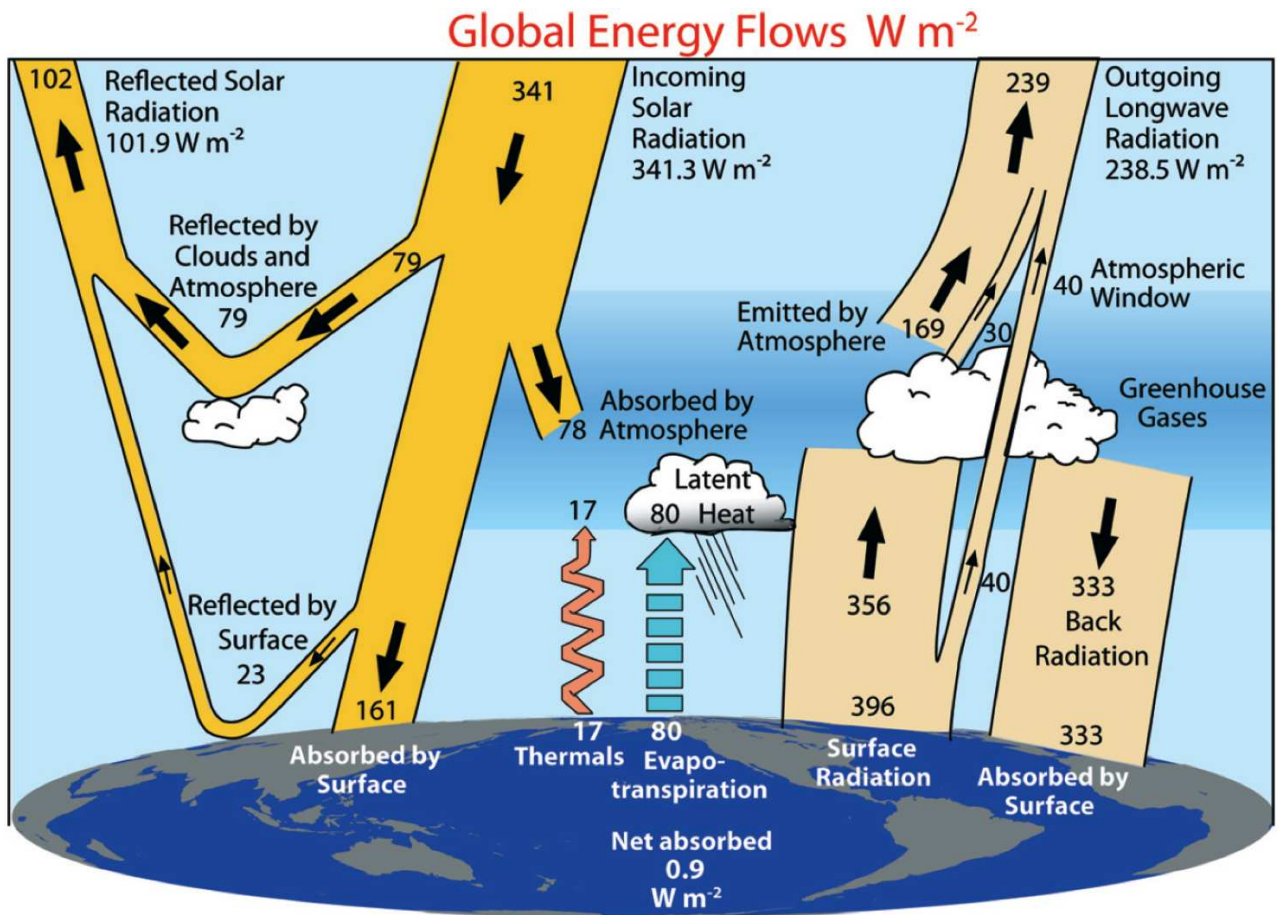


Figure 2: An updated version of Figure 1. Note that atmospheric absorption of short wave radiation is larger, and that a net absorbed amount of energy ($0.9 W/m^2$) has been "allowed" to account for the best estimate of actual climate energy imbalance. (From Trenberth et al. (2009)).

and other other trace gases gases such as O_3 . Also aerosols contribute directly to the greenhouse effect, although this contribution is very small and much smaller than the direct effect of aerosols on α . Indirectly, via clouds, aerosols are of cause important for the strength of the greenhouse effect. According to Figure 1 the total atmosphere - or greenhouse - effect is about $390 - 235 = 155 \text{ W/m}^2$.

Figure 2 provides an update of the numbers in Figure 1. There are some interesting differences. First of all the shortwave absorption in the atmosphere is increased considerably. The estimates of precipitation (i.e. also Evaporation) have also increased. These main changes have indirect implications for the other numbers which are all modified more or less.

So far we have just considered the present day situation. What happens if the climate starts to warm up due to increased solar irradiance, i.e. S increases? Basically (4) continues to apply, but we have to be aware that both α and δT depend on the actual climate state, i.e. T_s . Therefore we re-formulate the climate model as follows

$$\frac{S(1 - \alpha(T_s))}{4} - \sigma(T_s - \delta T(T_s))^4 = 0 \quad . \quad (5)$$

The α and δT dependencies on T_s describe short and long wave feedbacks, respectively, in the climate system. For example the planetary albedo α will increase if T_s decreases from its present value of 288 K. This is due to increased ice and snow covered areas. Similarly, δT decreases when it becomes colder, mainly because a colder atmosphere can hold less water vapour, which is the dominating greenhouse gas. The opposite applies when T_s increases: δT increases and α decreases, although the influence on α vanishes at some point when all snow and ice has melted.

Since both α and δT depend on the amount and type of cloud cover, which again depends on T_s , all cloud feedbacks are supposed to be included in the α and δT functions. In general, the functional influence of T_s on α and δT involves a huge number of climate feedback processes which can enhance or damp the initial change in T_s .

In analyses of climate processes it is customary to introduce the concept of radiative forcing, i.e. an imposed imbalance of the planetary radiation, which is due to some process that is not an inherent part of the climate system. Here we will distinguish between radiative forcings that are related to changes in the solar irradiance, and radiative forcings, F , of terrestrial origin. The global energy balance model can then be re-written once again:

$$F + \frac{S(1 - \alpha(T_s))}{4} - \sigma(T_s - \delta T(T_s))^4 = 0 \quad . \quad (6)$$

Note: a change in S of 1 Wm^{-2} corresponds to a global energy imbalance (radiative forcing) of $(1 - \alpha)/4 \text{ Wm}^{-2}$.

The terrestrial radiative forcing, F , can be due to geological and biological processes⁴, and anthropogenic activities. Geological processes can change the composition of the atmosphere, e.g. the CO_2 concentration, and the surface characteristics of the Earth. On short time scales, the strongest natural contribution to F comes from volcanic eruptions which emit huge amounts

⁴If the climate system is considered to consist of the entire Earth system the natural geological and biological contributions to F disappear and become part of the feedbacks. I.e. in a simple Earth System model they will be included in α and δT .

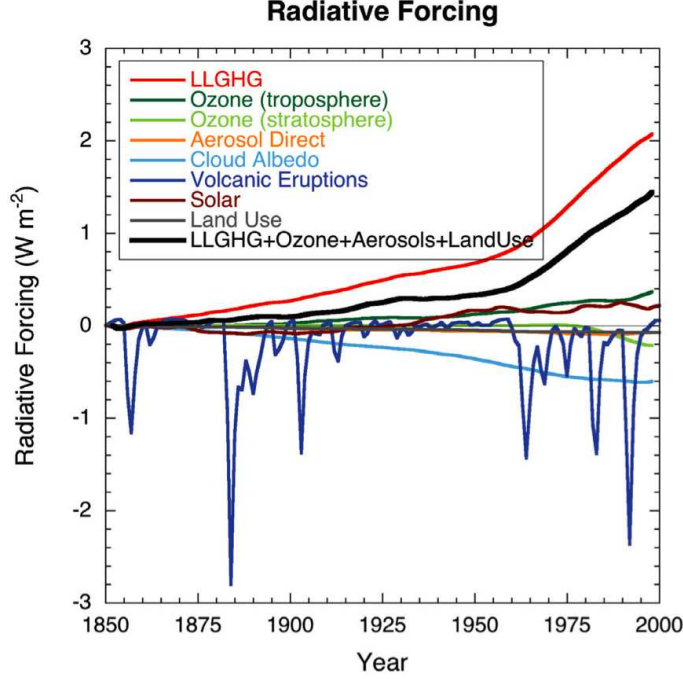


Figure 3: An estimate of the globally and annually averaged temporal evolution of the instantaneous radiative forcing due to various agents, as simulated in the MIROC+SPRINTARS model (Nozawa et al. (2005); Takemura and et al. (2005)).

of gases and particles into the atmosphere. In particular the sulphate (SO_2) emissions from explosive volcanoes can lead to formation of aerosols in the stratosphere which in a year or two after the eruption leads to a strong increase of the planetary albedo. The largest eruption in recent years, resulting in a short term (about one year) negative F -value with a peak value between -2 and -3 W/m^2 , was Pinatubo at the Philippines in 1991⁵. The total anthropogenic greenhouse gas contribution to the F value (since pre-industrial times) has been about 2.5 W/m^2 . A main contribution to this number comes from the emission of CO_2 and other greenhouse gases. Note, that we have chosen to include the radiative forcing related to anthropogenic greenhouse gases in the term F , although it would make more physical sense to implement such a forcing as a change in the function $\delta T(T_s)$.

A positive radiative forcing, i.e. $F > 0$, implies that T_s must increase a certain amount in order to satisfy (6). However, the required increase is highly sensitive to the above mentioned climate system feedbacks: e.g. the more steeply δT increases with T_s the larger the increase in T_s has to be.

As a final step we can make the global energy balance model time-dependent:

$$C \frac{\partial T_s}{\partial t} = F + \frac{S(1 - \alpha(T_s))}{4} - \sigma(T_s - \delta T(T_s))^4 \quad . \quad (7)$$

Here the left hand side represents the change per time unit of the heat content in a vertical column of water with an area of 1 m^2 and a heat capacity $C = d\rho_w L_w$, where d is the depth of the water column, ρ_w the density of water and L_w the specific heat of water. The depth d determines the

⁵Exersize: Calculate the change in planetary albedo, which, in terms of planetary radiative imbalance, would correspond to a radiative forcing of $F = -2.5 \text{ W/m}^2$.

thermal inertia of the model, i.e. how long time it takes to reach a new equilibrium after some terrestrial radiative forcing F , or a change in the solar constant, has been introduced.

3 Description of the meridional energy balance model

This section describes the meridional (south-north) one-dimensional model corresponding to the zero-dimensional model in (7), i.e. the model simulates the temporal evolution of the zonal mean⁶ surface temperature T_s at different latitudes. For simplicity we have introduced a coordinate x in the meridional direction which is equal to sine of the latitude ϕ , i.e. $x = \sin(\phi)$. One of the advantages obtained by this choice is that the surface area of the Earth between two x -values x_1 and x_2 is equal to $2a^2\pi(x_2 - x_1)$, assuming the Earth has a pure spherical shape.

If we only consider so-called local radiative balance/imbalance, i.e. the local surface temperature is determined by short wave input, long wave output and thermal inertia we obtain:

$$C \frac{\partial T_s(x, t)}{\partial t} = F + Q(x)(1 - \alpha(T_s(x, t))) - \sigma(T_s(x, t) - \delta T(T_s(x, t)))^4, \quad (8)$$

where $Q(x) = 0.25S(1 - 0.241(3x^2 - 1))$ is an approximation to the annual average downward short wave flux at the top of the atmosphere (TOA) at the present day inclination of the rotation axis of the Earth⁷.

Note that from now on all dependent variables listed below are functions of x and t , as opposed to the previous section which only considered global mean quantities. The forcing F in (12) is, however, introduced simply as a global mean forcing.

To obtain an estimate of the albedo, which agrees reasonably well with satellite based present day estimates, we use the expression⁸:

$$\alpha(T_s) = \min[0.7, \alpha_a + \alpha_s(T_s) - \alpha_a\alpha_s(T_s) - A_a\alpha_s(T_s)] \quad (12)$$

⁶The zonal mean is the average along a latitude circle

⁷Note that x varies between -1 (the South Pole) and 1 (the North Pole) and that the global integral of a variable Ψ depending only on latitude (such as the zonal mean of a variable) can be written as:

$$2\pi a^2 \int_{-\pi/2}^{\pi/2} \Psi \cos \phi d\phi = 2\pi a^2 \int_{-1}^1 \Psi dx. \quad (9)$$

As an example the globally integrated downward solar radiation at the top of the atmosphere becomes:

$$\begin{aligned} 2\pi a^2 \int_{-\pi/2}^{\pi/2} Q(\phi) \cos \phi d\phi &= 2\pi a^2 \int_{-1}^1 Q(x) dx \\ &= \frac{S}{2} \pi a^2 \int_{-1}^1 (1 - 0.241(3x^2 - 1)) dx = S\pi a^2 \end{aligned} \quad (10)$$

This is exactly the solar constant times the area of the Earth disc as seen from the sun.

If we introduce the notation $\langle \psi \rangle$ to indicate the global average of a variable ψ that is independent of longitude we get

$$\begin{aligned} \langle \psi \rangle &= \frac{2\pi a^2}{4\pi a^2} \int_{-1}^1 \psi(x) dx \\ &= \frac{1}{2} \int_{-1}^1 \psi(x) dx \end{aligned} \quad (11)$$

⁸The form of (12) is obtained in the following approximate way. Assume we have an atmospheric absorption

where $\alpha_a = \max[0.2 + 0.09x^2]$ and $\alpha_s(T_s) = 0.65f_i + (1 - f_i)(0.098 + 0.25x^4)$ represent the atmospheric (i.e. clouds, gases and particles) and surface (e.g. ice and snow) albedoes, respectively, and $A_a = 0.274(1 - x^2)$ is an estimate of the fractional atmospheric absorption of solar radiation. The parameter $f_i = k_1(273 - T_s)$ is the fraction of snow/sea ice with albedo set equal to 0.65. Note, that f_i is enforced to be within the interval $[0,1]$. The constant k_1 is a user specified constant, which determines the sensitivity of f_i to the surface temperature T_s , i.e. k_1 determines the strength of the ice/snow albedo feedback. The default value of k_1 is 0.03 K^{-1} . Note that we have not introduced any short wave atmospheric feedback parameter representing possible (and likely) short wave cloud feedbacks. In (12), and in what follows, we have for brevity omitted the functional arguments (x, t) for T_s and x for α_a and A_a , respectively.

We express the functional dependency of δT on T_s , i.e. all types of long wave feedbacks, as a very simple linear relationship:

$$\delta T(T_s) = \delta T_0 + k_3(T_s - T_{00}) = (\delta T_0 - k_3 T_{00}) + k_3 T_s \quad (13)$$

where $\delta T_0 = 34.5\text{K}$, $T_{00} = 287.5\text{K}$ and k_3 is a user-specified constant defining the strength of the long wave feedbacks. Positive values of k_3 correspond to positive feedbacks (i.e. enhancing) and negative values to negative feedbacks (i.e. damping). The default value of k_3 is 0.55. Due to the simplicity of the expression in 13 we have to restrict the minimum value of $\delta T(T_s)$ since there will always be some small greenhouse effect left in the atmosphere due to well mixed trace gases, clouds and aerosols. This minimum value has crudely been set to 10K.

If one uses the model in (8) to estimate the surface temperature at different latitudes, one obtains much too high temperatures in the tropics and much too low temperatures at high latitudes. This is because the atmospheric and oceanic flows in the real world give rise to a large scale turbulent poleward transport of heat. Figure 4 show an estimate of these transports. A simple way to include such transports in our energy balance model is to assume that energy crossing a given latitude can be re-presented as a large scale turbulent eddy diffusion. Then the convergence of this transport, i.e. the local accumulation of heat is:

$$\frac{\partial}{\partial x} \left(D(1 - x^2) \frac{\partial T_s}{\partial x} \right) \quad (14)$$

where D is a heat diffusion coefficient. The derivation of (14) is given in the appendix.

It has been argued (Alexeev et al., 2005) that the meridional diffusion coefficient should be sensitive to the global mean temperature $\langle T_s \rangle$ because the moisture content in the atmosphere for a given temperature increase tends to rise more at low (warm) latitudes than at high (cold) latitudes. Therefore the meridional transport of moisture, i.e. latent heat, increases. Effectively,

fraction A_a , an atmospheric albedo α_a and a surface albedo α_s and consider a unit area of the globe. Then the amount of short wave radiation reaching the surface is $Q(x)(1 - \alpha_a - A_a)$. Because of the surface albedo only the fraction $(1 - \alpha_s)$ of this number will actually be absorbed at the surface, i.e. the total amount of absorbed radiation at the surface is $Q(x)(1 - \alpha_a - A_a)(1 - \alpha_s) = Q(x)(1 - \alpha_a - A_a - \alpha_s + \alpha_a\alpha_s + A_a\alpha_s) = Q(x)(1 - (\alpha_a + A_a + \alpha_s - \alpha_a\alpha_s - A_a\alpha_s))$. The amount of absorbed solar radiation absorbed in the atmosphere is similarly $Q(x)A_a$ so that adding the surface and the atmospheric absorptions we get the total amount of solar radiation absorbed in the Earth system : $Q(x)(1 - (\alpha_a + \alpha_s - \alpha_a\alpha_s - A_a\alpha_s))$. This means that the effective albedo is $\alpha_a + \alpha_s - \alpha_a\alpha_s - A_a\alpha_s$. Note, that in the above we have ignored second order effects of re-absorption/reflection of reflected light. The limitation to the value 0.7 in (12) is based on observations of present day albedo over Antarctica where the value is about 0.7

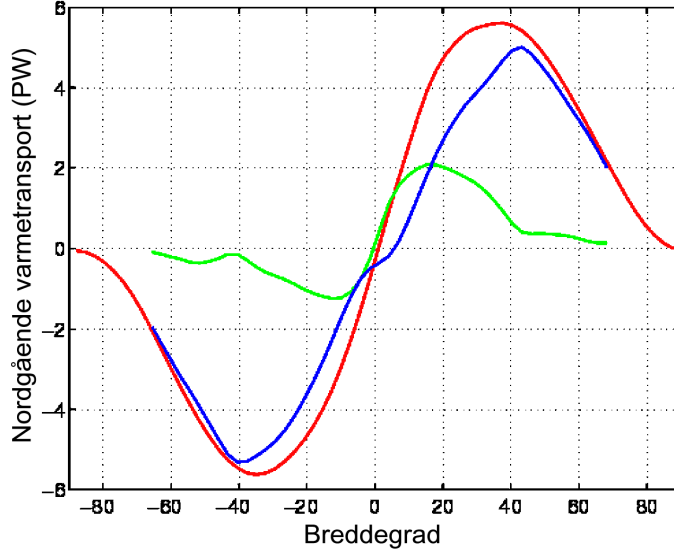


Figure 4: An estimate of the meridional heat transport. The red curve shows the transport needed to balance satellite based estimates of radiative imbalances at different latitudes. The blue curve is an estimate of the atmospheric transport based on the European Re-analysis project ERA15. The green curve is the residual (i.e. total minus atmospheric) representing the oceanic transport plus temporal storage. Units: PW = 10^{15} W

this represents an increase in the meridional heat transport. We therefore use the following expression to describe D

$$D = D_0 \max[0.5, 1. + k_2(\langle T_s \rangle - T_{00})] \quad (15)$$

The default values of D_0 and k_2 are $0.65 \text{ W/m}^2/\text{K}$ and 0.01 K^{-1} , respectively.

Using the expression in (14) to describe the meridional heat flux convergence we obtain the prognostic equation for our time dependent meridional energy balance model:

$$C \frac{\partial T_s}{\partial t} = F + Q(1 - \alpha(T_s)) - \sigma(T_s - \delta T(T_s))^4 + \frac{\partial}{\partial x} \left(D(1 - x^2) \frac{\partial T_s}{\partial x} \right) \quad (16)$$

The value of C has everywhere been set to $1.046 \times 10^9 \text{ J/K}$. This corresponds to a depth of the "water" of 250 m, and it gives an effective thermal inertia of the simple model which is in reasonable agreement with dynamical climate models.

4 Experimental setup

The model is initialised by a surface temperature profile which varies in the x -direction as follows:

$$T_s(x, 0) = T_0 - ax^2 + \int_{-1}^1 ax^2 dx = T_0 + a\left(\frac{2}{3} - x^2\right) \quad (17)$$

with $a = 45 \text{ K}$. T_0 is a user specified global mean initial surface temperature with default value of 287.5 K, i.e. 14.5 C. The model first reads the user specified constants if they are different from the default values. Then it simulates the evolution over 500 years with a user specified value of the present day solar constant $S = S_0$. The default value of S_0 is 1366 W/m^2 . This

Table 1: List of user specified values and their default values that are used if nothing (or "0") is specified.

Name	Description	Default value	Unit
k_1	Sensitivity of f_i to temperature	0.03	K^{-1}
k_2	Sensitivity of D to $\langle T_s \rangle$	0.01	K^{-1}
k_3	Sensitivity of local $T_s - T_e$ to T_s	0.55	
D_0	Present day turbulent diffusion coeff.	0.65	$\text{W}/\text{m}^2/\text{K}$
T_0	Initial value of $\langle T_s \rangle$	287.5	K
S_0	Present day/initial value of the solar constant	1366.0	W/m^2
S_1	Changed value of the solar constant	1366.0	W/m^2
F	Terrestrial radiative forcing	0.0	W/m^2

first simulation where the forcing parameter F is set to 0 is also termed "control simulation", to mimic the term used in dynamic climate modelling. The control simulation is followed by a climate change simulation where the solar constant is set to a different user specified value S_1 . In this second simulation the forcing parameter takes the user specified value F . Default values of S_1 and F are $1366 \text{ W}/\text{m}^2$ and $0.0 \text{ W}/\text{m}^2$, respectively. If you would like to simulate the effects of a doubling of the atmospheric CO_2 concentration you should replace the F value by $3.9 \text{ W}/\text{m}^2$.

A list of user specified values and their default values can be found in table 1.

5 How to run the model

To run the model go to the address:

<http://gfy.ku.dk/~kaas/onedmodel/run.php>

First type your name (or any user specific string). Then, after your own choice, type the user specified values in decimal form, including the period. Note that typing "0" (without a period) has the same effect as typing nothing! Also note that it may be necessary to "refresh" your browser if nothing seems to happen with the output graphics.

Press the "Run Program" button.

The model produces output which is presented both as graphs and as a table. You have to scroll down to see all the output. The graphs popping up are shown and described in Figure 5. The plots show various variables as indicated at the top of each panel. Note that the true latitude is given on the abscissa of the plots to enable comparison with other models.

In more detail the panels show the following graphs:

1. The initial, the present day simulated and the changed climate simulated temperatures (C) as functions of latitude. The heading of the panel provides information of the corresponding global mean averages.
2. Outgoing long wave radiation (OLR) in present day and changed climate simulations as function of latitude. Global mean values are shown in the heading. Units: W/m^2
3. Albedo in present day and changed climate simulations as function of latitude. The heading provides information of the global mean values.

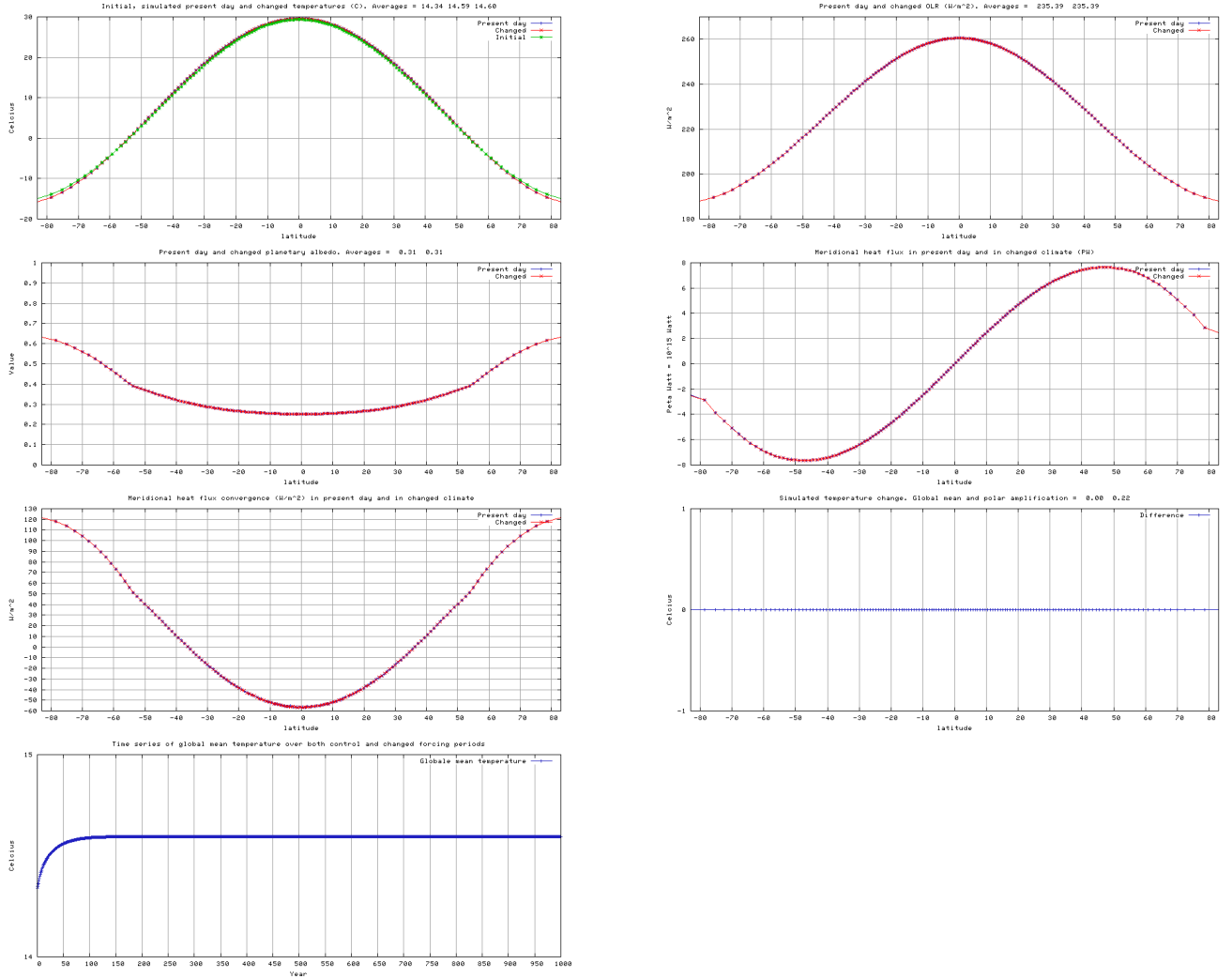


Figure 5: Example of the seven graphs that are shown once the model has been run. In this example the model was run with its default parameter values. A more detailed description of the individual panels is provided in the text.

4. The simulated present day and changed climate meridional heat fluxes in PW (10^{15} W)
5. Meridional heat flux convergence (W/m^2) in present day and in changed climate.
6. Simulated change in zonal mean temperature (C) as function of latitude. The degree of polar amplification, estimated as

$$(T_{s2}(1) - T_{s1}(1) - (\langle T_2 \rangle - \langle T_1 \rangle)) / (\langle T_2 \rangle - \langle T_1 \rangle)$$

is shown in the heading. Here subscripts "1" and "2" indicate the end of the control and the changed climate simulations, respectively.

7. Time series of global mean temperature (Units: C).

6 Exercises

Below a few exercises are suggested.

EXERCISE 1. Sensitivity of the model. Calculate the sensitivity of the model due to a doubling of the atmospheric concentration of CO₂ by running with a radiative forcing of 3.9 W/m². Express the result in K/(W/m²).

EXERCISE 2. Zero feedback sensitivity of the model. Estimate the zero feedback climate sensitivity by running the model with some forcing and with k_1 , k_2 and k_3 set to zero. Express the result in K/(W/m²).

EXERCISE 3. Gain parameter. Using the results from the two previous exercises, calculate the gain parameter of the model.

EXERCISE 4. The relative influence of different feedbacks. Run the model with a given forcing and only include one feedback, i.e. with only one of the parameters k_1 , k_2 and k_3 set at its default value and the remaining set to zero, varying this procedure for all feedbacks.

How does the theory of combining different feedbacks parameters into one fit the actual results.

EXERCISE 5. Change in solar constant, S . Identify what change in the solar constant is required to obtain the same climate change as with a doubling of the CO₂ concentration (i.e. alternatively running with a long wave radiative forcing of 3.9 W/m²). Estimate the change in the term $S(1 - \alpha)/4$ associated with the identified change in S , and compare with the forcing for 2 times CO₂ concentration (use a global mean α of 0.31).

EXERCISE 6. Change in the turbulent heat flux parameter D . What happens if D is set to zero? Vary D (but with k_2 set to zero) and explain why the changes in heat flux are not directly proportional to changes in D .

Fossiles of Crocodiles, originating from Eocene (53-57 million years ago), have been found on Ellesmere Island (now at 80N). Assuming that these creatures can survive if the annual mean temperature is 10C and that Ellesmere Island had approximately the same position as today (which is likely) estimate the value of D which is required to allow survival of the Crocodiles. What is the associated heat flux towards the poles.

EXERCISE 7. Polar amplification. Estimate the relative contribution in the model to polar amplification from sea-ice albedo effect and the meridional latent heat transport effect (run the model with a forcing of e.g. 4 W/m². Are the effects additive. Is there a polar amplification if both k_1 and k_2 are set to zero?.

EXERCISE 8. Sensitivity to initial condition. What happens if you set $T_0 = 240\text{K}$ and $k_2=0.0$. What happens if you set $T_0 = 245\text{K}$ and $k_2=0.0$ Explain the reasons for the difference.

7 Appendix. Derivation of the eddy heat flux convergence term

This appendix provides a derivation of the term

$$\frac{\partial}{\partial x} \left(D(1 - x^2) \frac{\partial T_s}{\partial x} \right), \quad (18)$$

which is the convergence of turbulent meridional heat flux in the atmosphere and the oceans. The parameter D is a heat diffusion coefficient with units $\text{W}/\text{m}^2/\text{K}$, T_s is surface temperature (i.e. also the temperature of the upper ocean) and $x = \sin \phi$, where ϕ is the latitude.

Let us start considering a small latitude interval from ϕ to $\phi + \delta\phi$ where $\delta\phi > 0$. The total northward heat flux over all longitudes across the latitude ϕ i.e. into the domain, can be approximated by

$$-2\pi a \cos(\phi) \left(\mathcal{D} \frac{\partial T_s}{a \partial \phi} \right) (\phi), \quad (19)$$

where \mathcal{D} is the "normally dimensioned" turbulent heat flux coefficient, which in general may depend on latitude, and a is the radius of the Earth. \mathcal{D} is equal to the northward energy flux per unit meter longitude per unit meridional temperature gradient, i.e it has units $(\text{W}/\text{m})/(\text{K}/\text{m}) = \text{W}/\text{K}$. Similarly the total northward heat flux over all longitudes across the latitude $\phi + \delta\phi$, i.e. out of the domain, can be approximated by

$$-2\pi a \cos(\phi + \delta\phi) \left(\mathcal{D} \frac{\partial T_s}{a \partial \phi} \right) (\phi + \delta\phi). \quad (20)$$

Note that in (19) and (20) the first factor $2\pi a \cos(\phi)$ and $2\pi a \cos(\phi + \delta\phi)$, respectively, are the distances around the Earth at each of the two latitudes ϕ and $\phi + \delta\phi$.

The net turbulent transport into the domain, i.e. the flux in minus the flux out, is

$$-2\pi a \cos(\phi) \left(\mathcal{D} \frac{\partial T_s}{a \partial \phi} \right) (\phi) + 2\pi a \cos(\phi + \delta\phi) \left(\mathcal{D} \frac{\partial T_s}{a \partial \phi} \right) (\phi + \delta\phi). \quad (21)$$

Now using first order Taylor series expansions we can approximate the terms $\cos(\phi + \delta\phi)$ and $(\mathcal{D} \partial T_s / (a \partial \phi))(\phi + \delta\phi)$ as

$$\cos(\phi + \delta\phi) \approx \cos(\phi) - \delta\phi \sin(\phi)$$

and

$$\left(\mathcal{D} \frac{\partial T_s}{a \partial \phi} \right) (\phi + \delta\phi) \approx \left(\mathcal{D} \frac{\partial T_s}{a \partial \phi} \right) (\phi) + \delta\phi \frac{\partial}{\partial \phi} \left(\mathcal{D} \frac{\partial T_s}{a \partial \phi} \right) (\phi).$$

Inserting in (21) we get the approximate expression for the net transport into the latitude band:

$$\begin{aligned} & -2\pi a \delta\phi \sin(\phi) \left(\mathcal{D} \frac{\partial T_s}{a \partial \phi} \right) (\phi) + 2\pi a \delta\phi \cos(\phi) \frac{\partial}{\partial \phi} \left(\mathcal{D} \frac{\partial T_s}{a \partial \phi} \right) (\phi) - 2\pi a (\delta\phi)^2 \sin(\phi) \frac{\partial}{\partial \phi} \left(\mathcal{D} \frac{\partial T_s}{a \partial \phi} \right) (\phi) \\ & \approx -2\pi \delta\phi \sin(\phi) \left(\mathcal{D} \frac{\partial T_s}{a \partial \phi} \right) (\phi) + 2\pi \delta\phi \cos(\phi) \frac{\partial}{\partial \phi} \left(\mathcal{D} \frac{\partial T_s}{a \partial \phi} \right) (\phi) \end{aligned} \quad (22)$$

where the last approximation is valid in the limit where $\delta\phi$ goes to zero. Now dividing (22) by the area on the globe represented by the latitude band (in the limit where $\delta\phi$ goes to zero this is approaches $2\pi a^2 \delta\phi \cos(\phi)$) we get the net turbulent heat flux convergence per unit area at latitude ϕ :

$$HC = -\mathcal{D} \frac{\tan(\phi)}{a} \frac{\partial T_s}{a \partial \phi} + \frac{\partial}{\partial \phi} \left(\mathcal{D} \frac{\partial T_s}{a \partial \phi} \right) \quad (23)$$

We can now perform the substitution $x = \sin(\phi)$ implying also that $\partial x = \cos(\phi)\partial\phi$. Using the following re-formulations:

$$-\mathcal{D}\frac{\tan(\phi)}{a}\frac{\partial T_s}{a\partial\phi} = -\frac{\mathcal{D}}{a^2}\sin(\phi)\frac{\partial T_s}{\cos(\phi)\partial\phi} = -\frac{\mathcal{D}}{a^2}x\frac{\partial T_s}{\partial x} \quad (24)$$

and

$$\begin{aligned} \frac{\partial}{a\partial\phi}\left(\mathcal{D}\frac{\partial T_s}{a\partial\phi}\right) &= \frac{\partial}{\partial\phi}\left(\frac{\mathcal{D}}{a^2}\cos(\phi)\frac{\partial T_s}{\cos(\phi)\partial\phi}\right) \\ &= \cos(\phi)\frac{\partial}{\partial\phi}\left(\frac{\mathcal{D}}{a^2}\frac{\partial T_s}{\cos(\phi)\partial\phi}\right) - \sin(\phi)\frac{\mathcal{D}}{a^2}\frac{\partial T_s}{\cos(\phi)\partial\phi} \\ &= \cos^2(\phi)\frac{\partial}{\cos(\phi)\partial\phi}\left(\frac{\mathcal{D}}{a^2}\frac{\partial T_s}{\cos(\phi)\partial\phi}\right) - \sin(\phi)\frac{\mathcal{D}}{a^2}\frac{\partial T_s}{\cos(\phi)\partial\phi} \\ &= (1-x^2)\frac{\partial}{\partial x}\left(\frac{\mathcal{D}}{a^2}\frac{\partial T_s}{\partial x}\right) - x\frac{\mathcal{D}}{a^2}\frac{\partial T_s}{\partial x} \end{aligned}$$

we can write (23) as:

$$\begin{aligned} HC &= (1-x^2)\frac{\partial}{\partial x}\left(\frac{\mathcal{D}}{a^2}\frac{\partial T_s}{\partial x}\right) - 2x\frac{\mathcal{D}}{a^2}\frac{\partial T_s}{\partial x} \\ &= \frac{\partial}{\partial x}\left(D(1-x^2)\frac{\partial T_s}{\partial x}\right) \end{aligned} \quad (25)$$

where $D = \mathcal{D}/a^2$.

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